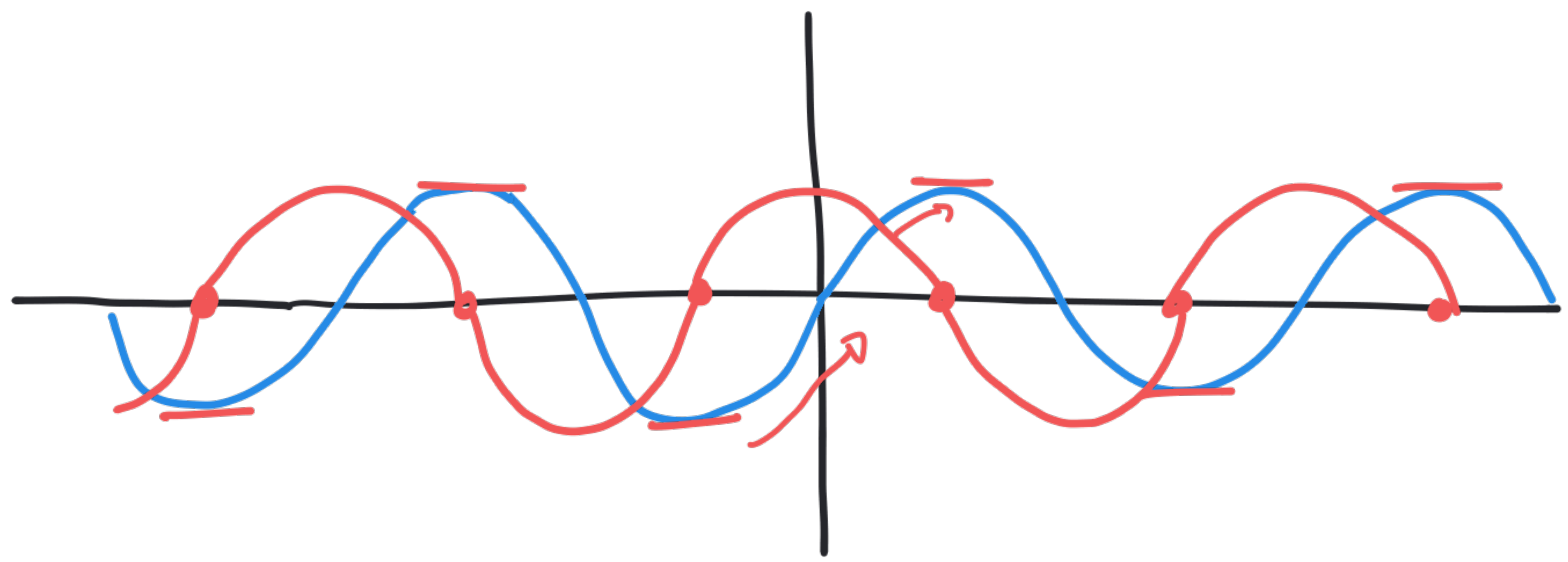
Intro Video: Section 3.3
Derivatives of Trigonometric Functions

Math F251X: Calculus 1

Derivatives of sin(x) & cos(x).



It looks like

$$\frac{d}{dx}(\sin(x)) = \cos(x)$$

How to prove this? Use the definition!

Identity:

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$

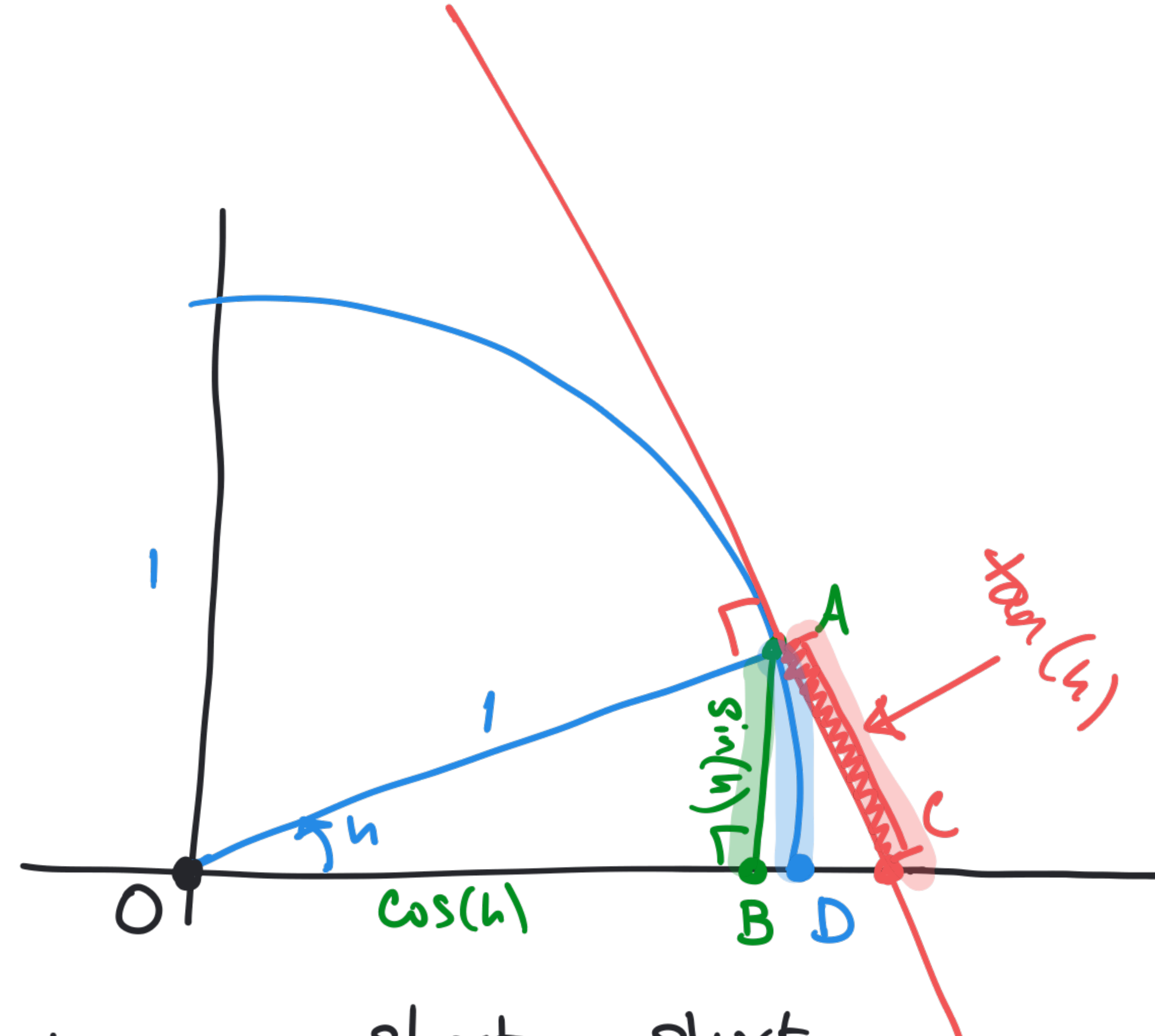
$$\frac{d}{dx}(\sin(x)) = \lim_{h \to 0} \frac{\boxed{\sin(x+h)} - \sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left(\sin(x)\cos(h) + \cos(x)\sin(h)\right) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x)\cos(h) - \sin(x)}{h} + \frac{\cos(x)\sin(h)}{h} = \lim_{h \to 0} \sin(x)\left(\frac{\cos(h)-1}{h}\right) + \lim_{h \to 0} \frac{\cos(x)\sin(h)}{h}$$

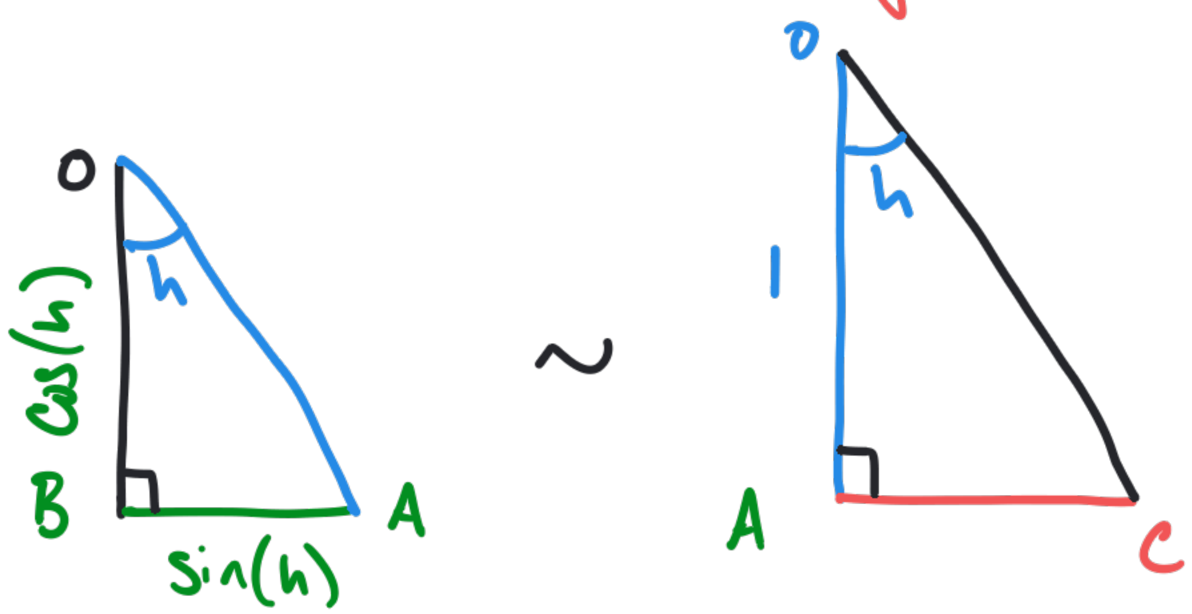
$$= \sin(x) \lim_{h \to 0} \frac{\cos(h)-1}{h} + \cos(x) \lim_{h \to 0} \frac{\sin(h)}{h}$$

What is $\lim_{h \to 0} \frac{\sin(h)}{h}$?



Claim: the length $AC = \tan(h)$.

Proof: Similar triangles!



Similar $\Delta \Rightarrow \frac{\text{short}}{\text{long}} = \frac{\text{short}}{\text{long}}$.

$\Rightarrow \frac{\sin(h)}{\cos(h)} = \frac{AC}{1} \Rightarrow AC = \tan(h)$.

FACT: Arclength $\overset{\frown}{AD} = h$

FACT: As long as h is small enough, $\boxed{\sin(h) < h < \tan(h)}$

So $\frac{\sin(h)}{h} < \frac{h}{h} < \frac{\tan(h)}{h} \Rightarrow \frac{\sin(h)}{h} < 1$ and $1 < \frac{\tan(h)}{h} \Rightarrow$

$1 < \frac{\sin(h)}{h\cos(h)} \Rightarrow$

$\cos(h) < \frac{\sin(h)}{h}$.

So: $\boxed{\cos(h) < \frac{\sin(h)}{h} < 1} \Rightarrow$ By the SQUEEZE THEOREM

$\lim_{h \to 0} \frac{\sin(h)}{h} = 1$.

We're still trying to compute $\frac{d}{dx}(\sin(x))$.

We just showed $\lim_{h \to 0} \frac{\sin(h)}{h} = 1$.

FACT: You can use this result and fancy trigonometric identities to show $\lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0$

Previously, we had that

$$\frac{d}{dx}(\sin(x)) = \ldots = \sin(x) \lim_{h \to 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h \to 0} \frac{\sin(h)}{h}$$

$$= \sin(x)(0) + \cos(x)(1)$$

$$= \cos(x).$$

SUMMARY: $\boxed{\frac{d}{dx}(\sin(x)) = \cos(x)}$ and $\boxed{\frac{d}{dx}(\cos(x)) = -\sin(x)}$

$$\frac{d}{dx}\left(\cos(x)\right) = \lim_{h \to 0} \frac{\cos(x+h) - \cos(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cos(x)\left(\cos(h) - 1\right)}{h} - \lim_{h \to 0} \frac{\sin(x)\sin(h)}{h}$$

$$= \cos(x) \lim_{h \to 0} \frac{\cos(h) - 1}{h} - \sin(x) \lim_{h \to 0} \frac{\sin(h)}{h}$$

$$= \cos(x)(0) - \sin(x)(1)$$

$$= -\sin(x).$$

We know:

① $\frac{d}{dx}(\sin(x)) = \cos(x)$

② $\frac{d}{dx}(\cos(x)) = -\sin(x)$

You showed (using the quotient rule):

③ $\frac{d}{dx}(\tan(x)) = \frac{1}{(\cos(x))^2} = (\sec(x))^2$

④ $\frac{d}{dx}(\sec(x)) = \frac{d}{dx}\left(\frac{1}{\cos(x)}\right) = \sec(x)\tan(x)$

$= \frac{\cos(x)(0) - 1(-\sin(x))}{(\cos(x))^2} = \frac{+\sin(x)}{(\cos(x))^2} = \tan(x)\sec(x)$

⑤ $\frac{d}{dx}(\csc(x)) = \frac{d}{dx}\left(\frac{1}{\sin(x)}\right) = -\csc(x)\cot(x)$

$= \frac{\sin(x)(0) - (1)(\cos(x))}{(\sin(x))^2} = \frac{-\cos(x)}{(\sin(x))^2} = -\cot(x)\cdot\csc(x)$

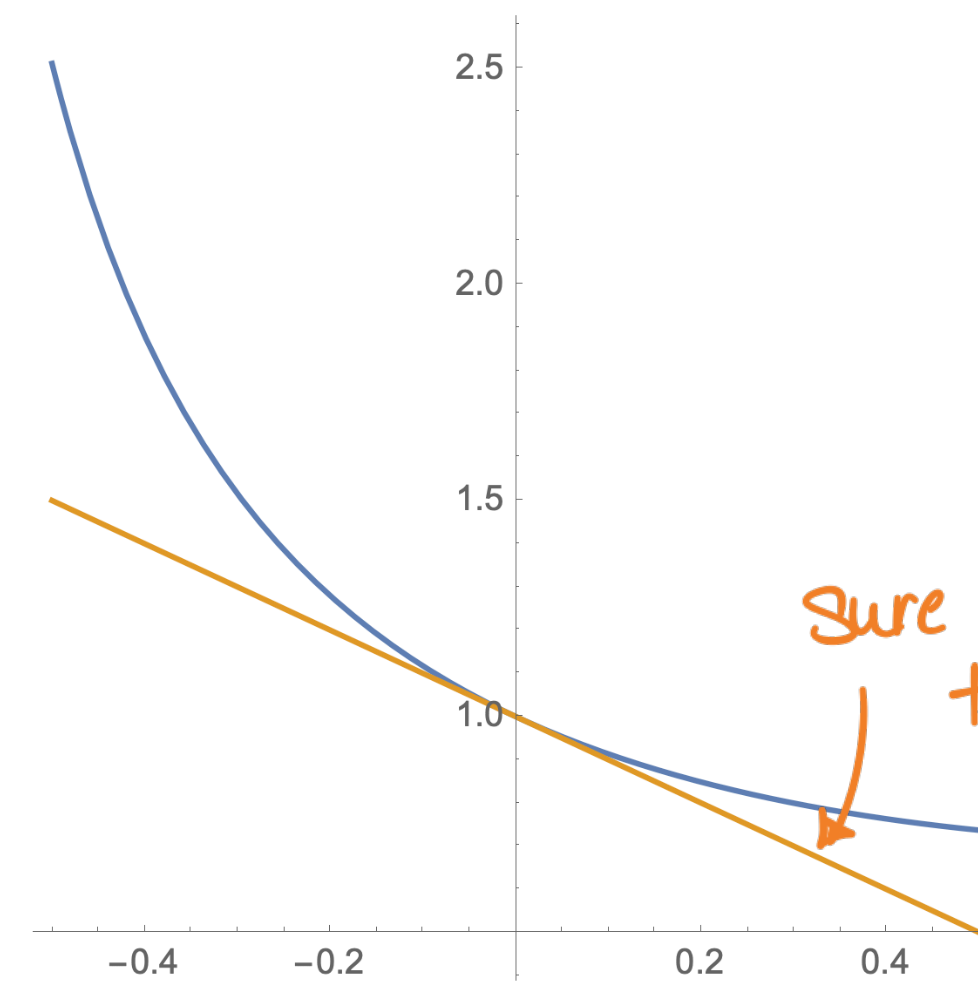
⑥ $\frac{d}{dx}(\cot(x)) = \frac{d}{dx}\left(\frac{\cos(x)}{\sin(x)}\right) = -(\csc(x))^2$

$= \frac{\sin(x)(-\sin(x)) - \cos(x)(\cos(x))}{(\sin(x))^2} = \frac{-((\sin(x))^2 + (\cos(x))^2)}{(\sin(x))^2} = \frac{-1}{(\sin(x))^2}$

Example: Find an equation for the tangent line to the graph of $y = \dfrac{1}{\sin(x)+\cos(x)}$ when $x=0$.

Here's our TL equation in the plot!

```
Plot[{1 / (Sin[x] + Cos[x]), -x + 1},
 {x, -1 / 2, 1 / 2}, AspectRatio → 1]
```



Sure looks tangent!

Need to find $\left.\dfrac{dy}{dx}\right|_{x=0}$

Evaluate $\dfrac{dy}{dx}$ at $x=0$

(If $y=f(x)$, this means find $f'(0)$.)

$$y' = \frac{(\sin(x)+\cos(x))(0) - (1)(\cos(x)-\sin(x))}{(\sin(x)+\cos(x))^2}$$

$$= \frac{\sin(x)-\cos(x)}{(\sin(x)+\cos(x))^2}.$$

So $y'(0) = \dfrac{\sin(0)-\cos(0)}{(\sin(0)+\cos(0))^2} = \dfrac{-1}{(0+1)^2} = -1$

And $y(0) = \dfrac{1}{\sin(0)+\cos(0)} = \dfrac{1}{0+1} = 1$

So TL has equation: $y = -1(x-0)+1 \Rightarrow y = -x+1$

<u>Example</u>: Suppose $f(\theta) = \theta \cdot \sin\theta$ Find the first four derivatives.

① $f'(\theta) = \dfrac{df}{d\theta} = \dfrac{d}{d\theta}(\theta \sin\theta) = \theta\cos(\theta) + \sin\theta$

② $f''(\theta) = \dfrac{d^2 f}{d\theta^2} = \dfrac{d}{d\theta}(\theta\cos\theta + \sin\theta)$

$= \theta(-\sin\theta) + \cos(\theta)(1) + \cos(\theta)$

$= -\theta\sin\theta + 2\cos\theta$

③ $f'''(\theta) = \dfrac{d^3 f}{d\theta^3} = \dfrac{d}{d\theta}(-\theta\sin\theta + 2\cos\theta)$

$= (-\theta)(\cos(\theta)) + \sin(\theta)(-1) + 2(-\sin\theta)$

$= -3\sin\theta - \theta\cos\theta$

④ $f^{(4)}(\theta) = \dfrac{d^4 f}{d\theta^4} = \dfrac{d}{d\theta}(-3\sin\theta - \theta\cos\theta) = -3\cos\theta - (\theta(-\sin\theta) + \cos\theta)$

$= -4\cos\theta + \theta\sin\theta$